

# Total decoupling of general quadratic pencils, Part I: Theory<sup>☆</sup>

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## Abstract

The notion of quadratic pencils,  $\lambda^2 M + \lambda C + K$ , where  $M$ ,  $C$ , and  $K$  are  $n \times n$  real matrices with or without some additional properties such as symmetry or positive definiteness, plays critical roles in many important applications. It has been long desirable, yet with very limited success, to reduce a complicated high-degree-of-freedom system to some simpler low-degree-of-freedom subsystems. Recently, Garvey, Friswell and Prells proposed a promising approach by which, under some mild assumptions, a general quadratic pencils can be converted by real-valued isospectral transformations into a totally decoupled system. This approach, if numerically feasible, would reduce the original  $n$ -degree-of-freedom second-order system to  $n$  totally independent single-degree-of-freedom second-order subsystems. Such a claim would be a striking breakthrough in the common knowledge that generally no three matrices  $M$ ,  $C$ , and  $K$  can be diagonalized simultaneously. This paper intends to serve three purposes: to clarify some of the ambiguities in the original proposition, to simplify some of the computational details and, most importantly, to complete the theory of existence by matrix polynomial factorization tactics.

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## 1. Introduction

The quadratic eigenvalue problem involves finding scalars  $\lambda$  and nonzero vectors  $\mathbf{u}$  to satisfy the equation

$$Q(\lambda)\mathbf{u} = 0, \quad (1)$$

where  $Q(\lambda)$  is the quadratic pencil

$$Q(\lambda) := Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K \quad (2)$$

defined by three given  $n \times n$  matrices  $M$ ,  $C$  and  $K$ . The scalars  $\lambda$  and the corresponding vectors  $\mathbf{u}$  are called, respectively, eigenvalues and eigenvectors of the quadratic pencil  $Q(\lambda)$ . It is known that the quadratic eigenvalue problem has  $2n$  finite eigenvalues over the complex field, provided that the leading matrix coefficient  $M$  is nonsingular. We shall consider only real-valued coefficient matrices in this paper.

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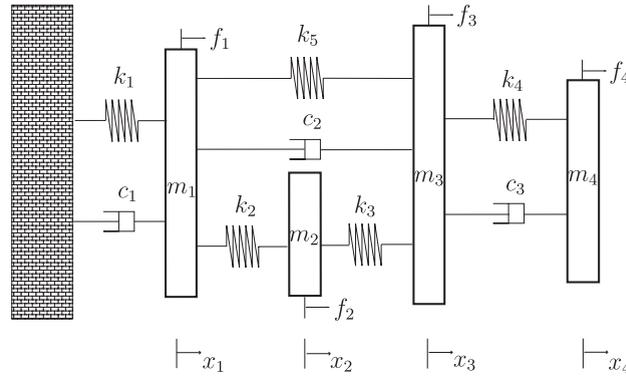


Fig. 1. A four-degree-of-freedom mass-spring system.

The quadratic eigenvalue problem has been studied extensively because its formation arises frequently in wide-ranging disciplines, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, and signal processing. In a recent treatise, Tisseur and Meerbergen [1] surveyed a good many applications, mathematical properties, and a variety of numerical techniques for the quadratic eigenvalue problem. The quadratic eigenvalue problem arising in practice often entails some additional conditions on the coefficient matrices. Consider, for example, the three-degree-of-freedom mass-spring system depicted in Fig. 1, where  $m_j$ ,  $c_j$ , and  $k_j$  represent the mass, damping and stiffness parameters, respectively. It is not difficult to see that the corresponding equation of motion has the following specifically structured matrix coefficients:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 & 0 & c_2 + c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 + k_5 & -k_2 & -k_5 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3 + k_4 + k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \mathbf{x} = \mathbf{f}(t).$$

In general, the second-order dynamical system with  $n$ -degree-of-freedom is of the form

$$M\ddot{\mathbf{x}} + (C + G)\dot{\mathbf{x}} + (K + N)\mathbf{x} = F(t), \tag{3}$$

where the coefficient matrices may have the following interpretations and structures:

$$\begin{cases} M := \text{Mass matrix}; & M = M^T \succ 0, \\ C := \text{Damping matrix}; & C = C^T, \\ K := \text{Stiffness matrix}; & K = K^T \succ 0, \\ G := \text{Gyroscopic matrix}; & G^T = -G, \\ N := \text{Dissipation matrix}; & N^T = -N, \\ F := \text{External force}. \end{cases}$$

It is the homogeneous solution of Eq. (3) that is most critical in the long-term behavior of the system. Assume the homogeneous solution  $\mathbf{x}(t)$  in the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{u}.$$

Upon substitution (and without causing ambiguity, denoting the linear and the constant coefficients by the same notation  $C$  and  $K$ , respectively), we find that  $\lambda$  and  $\mathbf{u}$  are the nontrivial solution to the quadratic eigenvalue problem:

$$Q(\lambda)\mathbf{u} := (\lambda^2 M + \lambda C + K)\mathbf{u} = 0.$$

An undamped quadratic eigenvalue problem, that is, when  $C = 0$ , is equivalent to a generalized eigenvalue problem of which the spectral analysis is well established. In particular, there are usually modal coordinates under which the undamped quadratic eigenvalue problem can be represented by diagonal coefficient matrices. In other words, the undamped quadratic eigenvalue problem can be totally decoupled. This reduction of the original multiple-degree-of-freedom system to totally decoupled single-degree-of-freedom second-order systems is very desirable from a practical point of view. Considerable efforts have been given to study the simultaneous diagonalization of two matrices by congruence or equivalence transformations. See, for example, the discussion in the book [2, Section 4.5] and the paper [3]. In a more realistic environment, however, it is more often that the quadratic eigenvalue problem is damped. It is therefore desired to generalize the notion of reduction to three matrices. In the literature, a quadratic pencil is said to be *proportionally* or *classically damped* if all three matrices  $M$ ,  $C$  and  $K$  can be diagonalized simultaneously. It is commonly accepted, however, that no equivalence or congruence coordinate transformations can simultaneously diagonalize the three coefficient matrices in a general quadratic system.

In two recent papers [4,5], Garvey et al. proposed a different way to perceive the simultaneous diagonalization of three matrices. We outline their ideas below. It is easy to show that the quadratic eigenvalue problem in Eq. (1) is equivalent to the generalized eigenvalue problem

$$L(\lambda) \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = 0, \tag{4}$$

where

$$L(\lambda) := L(\lambda; M, C, K) = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \lambda + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}. \tag{5}$$

Clearly, if  $M$  is nonsingular, then  $\mathbf{v} = \lambda\mathbf{u}$ . The arrangement in the *symmetrically* linearized pencil  $L(\lambda)$  is referred to as the *Lancaster structure*. If there exist nonsingular  $2n \times 2n$  matrices  $\Pi_\ell$  and  $\Pi_r$  such that the equivalence transformation applied to Eq. (5) maintains the Lancaster structure, that is,

$$\Pi_\ell L(\lambda) \Pi_r = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \lambda + \begin{bmatrix} K_D & 0 \\ 0 & -M_D \end{bmatrix} \tag{6}$$

and such that  $M_D$ ,  $C_D$  and  $K_D$  are all diagonal matrices, then the quadratic eigenvalue problem in Eq. (1) is equivalent to the totally decoupled system

$$(\lambda^2 M_D + \lambda C_D + K_D)\mathbf{z} = 0. \tag{7}$$

In this case, the eigenvectors  $\mathbf{u}$  and  $\mathbf{z}$  are related by

$$\begin{bmatrix} \mathbf{u} \\ \lambda\mathbf{u} \end{bmatrix} = \Pi_r \begin{bmatrix} \mathbf{z} \\ \lambda\mathbf{z} \end{bmatrix} \tag{8}$$

provided that  $M_D$  and  $M$  are nonsingular. Most importantly, the two quadratic systems in Eq. (1) and Eq. (7) are isospectral.

The notion outlined above is different from the usual task of simultaneously diagonalizing the coefficient matrices in the linearized pencil  $L(\lambda)$ . Rather, by maintaining the Lancaster structure, the approach links a multiple-degree-of-freedom system directly to a set of single-degree-of-freedom systems. For the idea to work, the following questions must be addressed:

- (1) Do the structure preserving transformations  $\Pi_\ell$  and  $\Pi_r$  exist?
- (2) Can the transformations  $\Pi_\ell$  and  $\Pi_r$  be real valued so that the resulting diagonal matrices  $M_D$ ,  $C_D$  and  $K_D$  remains to be real valued?

- (3) Is there any relationship between  $\Pi_\ell$  and  $\Pi_r$ , say,  $\Pi_\ell = \Pi_r^\top$ ?
- (4) How to find the real-valued transformations  $\Pi_\ell$  and  $\Pi_r$  numerically?

According to Garvey et al. [4], the answers to Questions 1 and 2 are affirmative. The proof was delineated in an appendix of Ref. [4] which, in our view, contains some ambiguities. We also think the instructions suggested in Ref. [4] for the construction of matrices  $\Pi_\ell$  and  $\Pi_r$  contain some errors and are unnecessarily complicated. The goal of this paper is to reconfirm the fact that the canonical form described in Eq. (6) is achievable by offering a clearer and simpler proof. Along the way, we can answer Question 3 which is an open problem speculated in Ref. [4]. We are able to advance to the completion of the theory by applying a useful notion from matrix polynomial factorization described in Ref. [6].

This paper contains two main parts: The first part addresses general pencils. In Section 3 we prove the existence of equivalence transformations by which almost all general quadratic pencils can be totally decoupled. The phrase “almost all” occurs several times in this presentation. It is a classical term used in mathematical analysis to reflect the fact that those elements which negate the underlying property in a statement form a set of Lebesgue measure zero in the ambient space. Our main contribution in this section is that some of the original arguments by Garvey et al. [4] are recharacterized in terms of rigorously defined index subsets, some of the ambiguities are cleared and the overall flow of logic is remarkably simplified. Assuming the availability of a complete spectral decomposition, the proof itself is constructive and can be converted into an algorithm. The second part addresses self-adjoint pencils. In Section 4 we begin with an outline of matrix polynomial factorization, showing that the real-valued eigenvalues of self-adjoint quadratic pencils are necessarily divided into two categories. We then show that the total decoupling can be achieved by congruence transformations. We believe these results are innovative and the theory of total decoupling is now complete.

Needless to say, it would be of great theoretical and practical significance if almost all  $n$ -degree-of-freedom systems can be completely decoupled into  $n$ -single-degree-of-freedom subsystems. This paper establishes the theoretical foundation that such a transformation is possible.

## 2. Nonlinear relationship

It has to be made clear that the procedure offered either in this paper or from Ref. [4] begins with the spectral decomposition of the pencil  $L(\lambda)$ , so the proof itself cannot serve as a satisfactory numerical means to answer Question 4. We do have a numerical way of working on the triplet  $(M, C, K)$  to reduce it to the triplet  $(M_D, C_D, K_D)$ , but the details will have to be discussed in a separate paper Ref. [7]. See also Ref. [8]. It is worth noting that the isospectral transformation from the triplet  $(M, C, K)$  to the triplet  $(M_D, C_D, K_D)$  is not the conventional equivalence transformation. Rather, it is a nonlinear relationship among all three matrices  $(M, C, K)$ .

Indeed, denoting

$$\Pi_\ell = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}, \quad \Pi_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \tag{9}$$

where each  $\ell_{ij}$  or  $r_{ij}$  is an  $n \times n$  matrices, in order to maintain the Lancaster structure in the product  $\Pi_\ell L(\lambda) \Pi_r$  it is necessary that the following five equations hold:

$$-\ell_{11}Kr_{12} + \ell_{12}Mr_{22} = 0,$$

$$-\ell_{21}Kr_{11} + \ell_{22}Mr_{21} = 0,$$

$$\ell_{21}Cr_{12} + \ell_{22}Mr_{12} + \ell_{21}Mr_{22} = 0,$$

$$\begin{aligned} \ell_{11}Cr_{12} + \ell_{12}Mr_{12} + \ell_{11}Mr_{22} &= \ell_{21}Cr_{11} + \ell_{22}Mr_{11} + \ell_{21}Mr_{21} \\ &= -\ell_{21}Kr_{12} + \ell_{22}Mr_{22}. \end{aligned} \tag{10}$$

Additionally, we are seeking  $\Pi_\ell$  and  $\Pi_r$  so that

$$\begin{aligned} -\ell_{21}Kr_{12} + \ell_{22}Mr_{22} &= M_D, \\ \ell_{11}Cr_{11} + \ell_{12}Mr_{11} + \ell_{11}Mr_{21} &= C_D, \\ \ell_{11}Kr_{11} - \ell_{12}Mr_{21} &= K_D, \end{aligned} \tag{11}$$

are diagonal matrices. The conditions in Eqs. (10) and (11) together constitute a nonlinear algebraic system of  $8n^2 - 3n$  equations in  $8n^2$  unknowns, but the system is not easy to solve.

The extra degrees of freedom in the underdetermined system Eqs. (10) and (11) suggest that both  $(M, C, K)$  and  $(M_D, C_D, K_D)$  reside on some nontrivial manifold. A structure preserving isospectral flow, that is, a differentiable path, starting from  $(M, C, K)$  is characterized in Ref. [8]. We shall describe a closed-loop feedback control system in the forthcoming paper [7] to drive such a flow to  $(M_D, C_D, K_D)$  numerically. The remaining of this paper shall concentrate on the theoretical issues.

### 3. General pencil and equivalence transformation

In this section, we detail steps toward proving the existence of the canonical form Eq. (6) for a quadratic pencil with general coefficient matrices  $M, C$  and  $K$  in  $\mathbb{R}^{n \times n}$ . For convenience, define the pair of matrices,

$$A := \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B := \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}. \tag{12}$$

Let  $(\lambda_j, \mathbf{x}_j), j = 1, \dots, 2n$ , denote the  $j$ th right eigenpair of the pencil  $\lambda B - A$ , that is, assume

$$A\mathbf{x}_j = \lambda_j B\mathbf{x}_j. \tag{13}$$

In general, the spectrum will be a mix of complex-valued and real-valued eigenvalues. Recall that the corresponding eigenvector  $\mathbf{u}_j$  of the original quadratic pencil  $Q(\lambda; M, C, K)$  can be recovered from the fact that

$$\mathbf{x}_j = \begin{bmatrix} \mathbf{u}_j \\ \lambda_j \mathbf{u}_j \end{bmatrix}. \tag{14}$$

To fix the idea, we shall begin with a spectral decomposition where the following scheme concerning real and complex eigenvalues and the associated eigenvectors holds:

- Each pair of complex conjugate eigenvalues and the corresponding eigenvectors is placed next to each other, and
- The eigenvectors corresponding to real-valued eigenvalues are real-valued.

This scheme will be changed along our later development. It will be convenient to adopt the notation,

$$A := \text{diag}\{\lambda_1, \lambda_2, \dots, \dots, \lambda_{2n}\} \in \mathbb{C}^{2n \times 2n} \tag{15}$$

and

$$X := [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n}] \in \mathbb{C}^{2n \times 2n}, \tag{16}$$

representing the eigenvalues and the eigenvectors, respectively. Likewise, let  $(\lambda_j, \mathbf{y}_j), j = 1, \dots, 2n$ , denote the  $j$ th left eigenpair of the pencil  $\lambda B - A$ , that is, assume

$$\mathbf{y}_j^\top A = \lambda_j \mathbf{y}_j^\top B. \tag{17}$$

Denote the corresponding matrix of left eigenvectors by  $Y \in \mathbb{C}^{2n \times 2n}$ . It is clear that when  $A$  and  $B$  are both symmetric, we can choose  $Y = X$ . This point will be stressed in Section 4 for self-adjoint pencils. Be aware that we use “transpose” rather than “conjugate transpose” for the left eigenvectors.

For simplicity, we shall assume henceforth that all eigenvalues of  $L(\lambda)$  are simple, that is, we shall assume that the diagonal matrix  $A$  has distinct diagonal entries. We think our argument below can be generalized in a

straightforward yet tedious way to the case where nontrivial Jordan chains occur, but we shall not elaborate the details in this paper. We shall assume further that  $A$  is invertible. The following argument can easily be modified for the case if 0 is a simple eigenvalue of  $L(\lambda)$ .

Observe from Eqs. (13) and (17) that the relationship

$$Y^T B X \Lambda = \Lambda Y^T B X = Y^T A X \tag{18}$$

holds. The first equality in Eq. (18) indicates that  $Y^T B X$  commutes with the diagonal matrix  $\Lambda$  which has distinct entries. It follows that the two matrices  $A_1$  and  $B_1$  defined by

$$\begin{aligned} A_1 &:= Y^T A X, \\ B_1 &:= Y^T B X \end{aligned} \tag{19}$$

must also be diagonal. We make the general assumption that the leading matrix coefficient  $M$  is nonsingular. It follows that  $B_1^{-1}$  exists. Clearly, the scaled columns of the matrices,

$$\begin{aligned} X^{[2]} &:= X B_1^{-1/2} A^{-1/2}, \\ Y^{[2]} &:= Y B_1^{-1/2} A^{-1/2}, \end{aligned} \tag{20}$$

where diagonal entries of  $B_1^{1/2}$  and  $A^{-1/2}$  are the principal square roots of those of  $B_1$  and  $A$ , respectively, are still the right and left eigenvectors of  $L(\lambda)$ , respectively. By using this set of scaled eigenvectors, we see that

$$\begin{aligned} A_2 &:= Y^{[2]T} A X^{[2]} = I_{2n}, \\ B_2 &:= Y^{[2]T} B X^{[2]} = A^{-1}, \end{aligned} \tag{21}$$

where  $I_{2n}$  stands for the  $2n \times 2n$  identity matrix. In order to achieve Eq. (21), it is important to note that the scaled eigenvectors  $\mathbf{x}_i^{[2]}$  and  $\mathbf{y}_i^{[2]}$  corresponding to the real eigenvalues  $\lambda_i$  can become purely imaginary, if  $\mathbf{y}_i^T A \mathbf{x}_i < 0$ .

### 3.1. Classification

It is useful to identify the special characteristics of eigenvalues by their locations in the spectrum. We shall use these locations as pointers in the diagonalization process.

Let  $\mathfrak{C}$ ,  $\text{Re}$  and  $\text{Im}$  denote, respectively, the index (locality) sets of complex-valued, real-valued, and purely imaginary-valued eigenvectors in  $X^{[2]}$ . It is possible that some, but not all, such index sets are empty. Let elements in  $\mathfrak{C}$  be denoted by  $c_j, j = 1, \dots, |\mathfrak{C}|$ , where  $|\mathfrak{C}|$  stands for the cardinality of the set  $\mathfrak{C}$ . Likewise, let elements in  $\text{Re}$  and  $\text{Im}$  be denoted by  $r_j$  and  $i_j$ , respectively. Write

$$\mathfrak{C} = \mathfrak{C}^+ \cup \mathfrak{C}^-,$$

where  $\mathfrak{C}^+$  and  $\mathfrak{C}^-$  are the index sets of complex-valued eigenvalues with positive and negative imaginary parts, respectively. Let  $p := |\mathfrak{C}^+|$  and  $\rho := \min\{|\text{Re}|, |\text{Im}|\}$ . We now further regroup the *real eigenvalues* into six subcategories according to the following rules which are described in Ref. [4]:

- (1) If the end index is less than the start index in any of the following “for” statements, define the corresponding set to be empty.
- (2) For  $j = 1, \dots, \rho$ ,

$$\begin{aligned} \{r_j, i_j\} \in C_a &\Leftrightarrow \lambda_{r_j} \lambda_{i_j} > 0, \\ \{r_j, i_j\} \in C_f &\Leftrightarrow \lambda_{r_j} \lambda_{i_j} < 0. \end{aligned}$$

- (3) For  $j = 1, \dots, |\text{Re}| - \rho/2$ ,

$$\{r_{\rho+2j-1}, r_{\rho+2j}\} \in C_b \Leftrightarrow \lambda_{r_{\rho+2j-1}} \lambda_{r_{\rho+2j}} < 0,$$

$$\{\mathbf{r}_{\rho+2j-1}, \mathbf{r}_{\rho+2j}\} \in C_d \Leftrightarrow \lambda_{\mathbf{r}_{\rho+2j-1}} \lambda_{\mathbf{r}_{\rho+2j}} > 0.$$

(4) For  $j = 1, \dots, |\text{Im}| - \rho/2$ ,

$$\{\mathbf{i}_{\rho+2j-1}, \mathbf{i}_{\rho+2j}\} \in C_c \Leftrightarrow \lambda_{\mathbf{i}_{\rho+2j-1}} \lambda_{\mathbf{i}_{\rho+2j}} < 0,$$

$$\{\mathbf{i}_{\rho+2j-1}, \mathbf{i}_{\rho+2j}\} \in C_e \Leftrightarrow \lambda_{\mathbf{i}_{\rho+2j-1}} \lambda_{\mathbf{i}_{\rho+2j}} > 0.$$

Note that each category  $C_v$ ,  $v = a, b, \dots, f$ , contains simply indices of real eigenvalues. Note also that categories  $C_a$  and  $C_f$ , and similarly others, are in some sense “complementary” to each other. By construction, these indices appear in pairs. Collect all first indices from these pairs in  $C_v$  into a subset  $C_v^+$  and all second indices into  $C_v^-$ . We have

$$C_v = C_v^+ \cup C_v^-.$$

Denote  $q_v := |C_v|/2 = |C_v^+| = |C_v^-|$ . For convenience, we further collect the indices into two sets:

$$\mathfrak{I}^+ := [C_a^+, \dots, C_f^+],$$

$$\mathfrak{I}^- := [C_a^-, \dots, C_f^-].$$

Define the permutation matrix  $P \in \mathbb{R}^{2n \times 2n}$  by rearranging the columns of the identity matrix  $I_{2n}$  according to the index vector

$$\delta = [\mathfrak{C}^+, \mathfrak{I}^+, \mathfrak{C}^-, \mathfrak{I}^-]. \tag{22}$$

Note that  $|\mathfrak{I}^+| = |\mathfrak{I}^-| = n - p$ . By rearranging the eigenvectors accordingly,

$$X^{[3]} := X^{[2]}P = XB_1^{-1/2}A^{-1/2}P,$$

$$Y^{[3]} := Y^{[2]}P = YB_1^{-1/2}A^{-1/2}P, \tag{23}$$

we obtain from Eq. (21)

$$A_3 := Y^{[3]\top}AX^{[3]} = I_{2n},$$

$$B_3 := Y^{[3]\top}BX^{[3]} = \begin{bmatrix} A_{\mathfrak{C}^+}^{-1} & 0 & & & & \\ & 0 & A_{\mathfrak{I}^+}^{-1} & & & \\ & & & A_{\mathfrak{C}^-}^{-1} & 0 & \\ & & & 0 & A_{\mathfrak{I}^-}^{-1} & \end{bmatrix}, \tag{24}$$

where each diagonal block in  $B_3$  is composed of the eigenvalues from the indicated subset of indices. Note that generally  $X^{[3]}$  are  $Y^{[3]}$  are complex-valued. We remark that the eigenvectors in  $X^{[3]}$  (and, likewise, those of  $Y^{[3]}$ ) have been rearranged in such a way that

$$X^{[3]} = [X_{\mathfrak{C}^+}^{[3]}, X_{\mathfrak{I}^+}^{[3]}, X_{\mathfrak{C}^-}^{[3]}, X_{\mathfrak{I}^-}^{[3]}].$$

For example,

$$X_{\mathfrak{I}^+}^{[3]} = [X_{C_a^+}^{[3]}, X_{C_b^+}^{[3]}, X_{C_c^+}^{[3]}, X_{C_d^+}^{[3]}, X_{C_e^+}^{[3]}, X_{C_f^+}^{[3]}],$$

where columns in  $X_{C_a^+}^{[3]}$ ,  $X_{C_b^+}^{[3]}$ ,  $X_{C_d^+}^{[3]}$ , and  $X_{C_f^+}^{[3]}$  are real values while those in  $X_{C_c^+}^{[3]}$  and  $X_{C_e^+}^{[3]}$  purely imaginary.

Up to now we have been following along the procedures suggested in Ref. [4], except that we are providing more specific classifications by the indices. Now we begin to deviate. For each  $v = a, b, \dots, f$ , let  $E_v$

denote the  $q_v \times q_v$  identity matrix  $I_{q_v}$ . Define the following four block diagonal matrices:

$$\begin{cases} L^+ := \text{diag}\{E_a, E_b, iE_c, E_d, -iE_e, E_f\}, \\ L^- := \text{diag}\{iE_a, E_b, iE_c, -E_d, iE_e, -iE_f\}, \\ R^+ := \text{diag}\{E_a, E_b, iE_c, E_d, iE_e, E_f\}, \\ R^- := \text{diag}\{iE_a, E_b, iE_c, E_d, iE_e, iE_f\}, \end{cases} \quad (25)$$

where  $i = \sqrt{-1}$ . It is important to note that columns in the products

$$\begin{aligned} [X_{\mathfrak{z}^+}^{[4]}, X_{\mathfrak{z}^-}^{[4]}] &:= [X_{\mathfrak{z}^+}^{[3]}, X_{\mathfrak{z}^-}^{[3]}] \begin{bmatrix} R^+ & 0 \\ 0 & R^- \end{bmatrix}, \\ [Y_{\mathfrak{z}^+}^{[4]}, Y_{\mathfrak{z}^-}^{[4]}] &:= [Y_{\mathfrak{z}^+}^{[3]}, Y_{\mathfrak{z}^-}^{[3]}] \begin{bmatrix} L^+ & 0 \\ 0 & L^- \end{bmatrix}, \end{aligned} \quad (26)$$

are all real-valued now. Observe the facts that

$$\begin{bmatrix} L^+ & 0 \\ 0 & L^- \end{bmatrix} \begin{bmatrix} R^+ & 0 \\ 0 & R^- \end{bmatrix} = \text{diag}\{\widehat{I}_{n-p}, \widetilde{I}_{n-p}\}, \quad (27)$$

where

$$\begin{cases} \widehat{I}_{n-p} := \text{diag}\{E_a, E_b, -E_c, E_d, E_e, E_f\}, \\ \widetilde{I}_{n-p} := \text{diag}\{-E_a, E_b, -E_c, -E_d, -E_e, E_f\} \end{cases} \quad (28)$$

and that both diagonal matrices  $\Omega^+$  and  $\Omega^-$  defined by

$$\begin{cases} \Omega^+ := L^+ A_{\mathfrak{z}^+}^{-1} R^+ = \text{diag}\{A_{C_a^+}^{-1}, A_{C_b^+}^{-1}, -A_{C_c^+}^{-1}, A_{C_d^+}^{-1}, A_{C_e^+}^{-1}, A_{C_f^+}^{-1}\}, \\ \Omega^- := L^- A_{\mathfrak{z}^-}^{-1} R^- = \text{diag}\{-A_{C_a^-}^{-1}, A_{C_b^-}^{-1}, -A_{C_c^-}^{-1}, -A_{C_d^-}^{-1}, -A_{C_e^-}^{-1}, A_{C_f^-}^{-1}\}, \end{cases} \quad (29)$$

remain real-valued.

We summarize the procedure thus far as follows:

**Lemma 1.** Define the realization matrices

$$J_\ell := \begin{bmatrix} \frac{1}{\sqrt{2}} I_p & 0 & \frac{-i}{\sqrt{2}} I_p & 0 \\ 0 & L^+ & 0 & 0 \\ \frac{1}{\sqrt{2}} I_p & 0 & \frac{i}{\sqrt{2}} I_p & 0 \\ 0 & 0 & 0 & L^- \end{bmatrix}, \quad J_r := \begin{bmatrix} \frac{1}{\sqrt{2}} I_p & 0 & \frac{-i}{\sqrt{2}} I_p & 0 \\ 0 & R^+ & 0 & 0 \\ \frac{1}{\sqrt{2}} I_p & 0 & \frac{i}{\sqrt{2}} I_p & 0 \\ 0 & 0 & 0 & R^- \end{bmatrix}, \quad (30)$$

then the matrices

$$\begin{aligned} X^{[4]} &:= X^{[3]} J_r = X B_1^{-1/2} A^{-1/2} P J_r, \\ Y^{[4]} &:= Y^{[3]} J_\ell = Y B_1^{-1/2} A^{-1/2} P J_\ell \end{aligned} \quad (31)$$

are real-valued. In this case, the pencil  $\lambda B - A$  can be transformed isospectrally into  $\lambda B_4 - A_4$ , where

$$A_4 := Y^{[4]\top} A X^{[4]} = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & \widehat{I}_{n-p} & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & \widetilde{I}_{n-p} \end{bmatrix},$$

$$B_4 := Y^{[4]\top} B X^{[4]} = \begin{bmatrix} \operatorname{Re}(A_{\mathbb{C}^+}^{-1}) & 0 & \operatorname{Im}(A_{\mathbb{C}^+}^{-1}) & 0 \\ 0 & \Omega^+ & 0 & 0 \\ \operatorname{Im}(A_{\mathbb{C}^+}^{-1}) & 0 & -\operatorname{Re}(A_{\mathbb{C}^+}^{-1}) & 0 \\ 0 & 0 & 0 & \Omega^- \end{bmatrix}, \tag{32}$$

with  $\operatorname{Re}$  and  $\operatorname{Im}$  standing for the real and the imaginary parts of a complex number, respectively.

We stress that  $\lambda B_4 - A_4$  is a real-valued pencil obtained by real-valued equivalence transformations. We need to further reduce  $A_4$  and  $B_4$  into the form characterized in Eq. (6).

### 3.2. Elimination

Our next step is to employ equivalence transformation by a matrix of the form

$$F := \begin{bmatrix} \Phi & 0 & I_p & 0 \\ 0 & \Psi^+ & 0 & I_{n-p} \\ I_p & 0 & \Phi & 0 \\ 0 & I_{n-p} & 0 & \Psi^- \end{bmatrix}, \tag{33}$$

where  $\Phi$ ,  $\Psi^+$  and  $\Psi^-$  are real-valued diagonal matrices, to eliminate the lower right  $n \times n$  block of  $B_4$  while maintaining the diagonal form of  $A_4$ . Since all blocks involved in Eqs. (32) and (33) are diagonal, it suffices to consider the elimination entry by entry. We consider the entries of  $\Phi$ ,  $\Psi^+$  and  $\Psi^-$  separately.

To define the  $k$ th diagonal entry  $\phi_k$  in  $\Phi$ , let  $\alpha_k + i\beta_k$  represent the  $k$ th diagonal entry in  $A_{\mathbb{C}^+}$ . Observe that

$$\begin{bmatrix} \phi_k & 1 \\ 1 & \phi_k \end{bmatrix} \begin{bmatrix} \alpha_k & \beta_k \\ \beta_k & -\alpha_k \end{bmatrix} \begin{bmatrix} \phi_k & 1 \\ 1 & \phi_k \end{bmatrix} = \begin{bmatrix} \alpha_k \phi_k^2 + 2\beta_k \phi_k - \alpha_k & \beta_k + \beta_k \phi_k^2 \\ \beta_k + \beta_k \phi_k^2 & \alpha_k + 2\beta_k \phi_k - \alpha_k \phi_k^2 \end{bmatrix}.$$

The (2, 2) entry of the  $2 \times 2$  matrix on the right-hand side above can be eliminated by choosing

$$\phi_k := \frac{\beta_k + \sqrt{\alpha_k^2 + \beta_k^2}}{\alpha_k}. \tag{34}$$

In matrix form, this choice can be written as

$$\Phi := (\operatorname{Im}(A_{\mathbb{C}^+}^{-1}) + \sqrt{(\operatorname{Re}(A_{\mathbb{C}^+}^{-1}))^2 + (\operatorname{Im}(A_{\mathbb{C}^+}^{-1}))^2} (\operatorname{Re}(A_{\mathbb{C}^+}^{-1}))^{-1}). \tag{35}$$

Likewise, let  $k$ th entries of  $\Omega^+$  and  $\Omega^-$  be denoted as  $\omega_k^+$  and  $\omega_k^-$ , respectively. Observe that

$$\begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} \begin{bmatrix} \omega_k^+ & 0 \\ 0 & \omega_k^- \end{bmatrix} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} = \begin{bmatrix} \omega_k^+ (\psi_k^+)^2 + \omega_k^- & \omega_k^+ \psi_k^+ + \omega_k^- \psi_k^- \\ \omega_k^+ \psi_k^+ + \omega_k^- \psi_k^- & \omega_k^+ + \omega_k^- (\psi_k^-)^2 \end{bmatrix}.$$

In order to eliminate the (2,2) entry, we must choose

$$\psi_k^- := \pm \sqrt{-\frac{\omega_k^+}{\omega_k^-}}. \tag{36}$$

One immediate concern is whether the right-hand side in Eq. (36) is a real number, but that concern is perfectly addressed in the specific way we choose signs when defining the four matrices  $L^\pm$  and  $R^\pm$ .

**Lemma 2.** *By the way the index subsets  $C_v$ ,  $v = a, b, \dots, f$  are defined and, subsequently, the diagonal matrices  $\Omega^+$  and  $\Omega^-$  are arranged,  $\omega_k^+$  and  $\omega_k^-$  have opposite signs.*

At its the first glance, the choice of sign in Eq. (36) does not seem important. But if we want to maintain the diagonal form when the same equivalence transformation is applied to  $A_4$ , the signs of  $\psi_k^+$  and  $\psi_k^-$  must be selected more carefully. The reason can be seen from the following two calculations:

$$\begin{aligned} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} &= \begin{bmatrix} (\psi_k^+)^2 + 1 & \psi_k^+ + \psi_k^- \\ \psi_k^+ + \psi_k^- & 1 + (\psi_k^-)^2 \end{bmatrix}, \\ \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} &= \begin{bmatrix} (\psi_k^+)^2 - 1 & \psi_k^+ - \psi_k^- \\ \psi_k^+ - \psi_k^- & 1 - (\psi_k^-)^2 \end{bmatrix}. \end{aligned}$$

Depending on whether the  $k$ th diagonal entry in the product  $\hat{I}_{n-p}\tilde{I}_{n-p}$  is a positive or negative one, we have to choose  $\psi_k^+ = -\psi_k^-$  or  $\psi_k^+ = \psi_k^-$ , accordingly, in order to keep the off-diagonal entries of the  $2 \times 2$  matrices on the right-hand side above zero. In matrix form, if we choose

$$\Psi^+ := \sqrt{-\Omega^+(\Omega^-)^{-1}}, \tag{37}$$

then

$$\Psi^- := -\hat{I}_{n-p}\tilde{I}_{n-p}\Psi^+. \tag{38}$$

The elimination process is summarized as follows:

**Lemma 3.** *From the real-valued matrix  $F$  in Eq. (33) with diagonal matrices  $\Phi$ ,  $\Psi^+$  and  $\Psi^-$  given by Eqs. (35), (37) and (38), respectively. Define*

$$\begin{aligned} X^{[5]} &:= X^{[4]}F = XB_1^{-1/2}A^{-1/2}PJ_rF, \\ Y^{[5]} &:= Y^{[4]}F = YB_1^{-1/2}A^{-1/2}PJ_\ell F, \end{aligned} \tag{39}$$

then the two matrices

$$\begin{aligned} A_5 &:= Y^{[5]\top}AX^{[5]} = \begin{bmatrix} A_5^{[11]} & 0 \\ 0 & A_5^{[22]} \end{bmatrix}, \\ B_5 &:= Y^{[5]\top}BX^{[5]} = \begin{bmatrix} B_5^{[11]} & B_5^{[12]} \\ B_5^{[12]} & 0 \end{bmatrix}, \end{aligned} \tag{40}$$

where all  $A_5^{[kj]}$  and  $B_5^{[kj]}$ ,  $j, k = 1, 2$ , are diagonal matrices, form a real-valued pencil isospectral to the original  $\lambda B - A$ .

### 3.3. Scaling

Our last step is to scale the pair  $(A_5, B_5)$  back to the Lancaster structure. This can be accomplished by a diagonal matrix of the form

$$\Gamma := \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix}. \tag{41}$$

Indeed, observe that

$$\begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} A_5^{[11]} & 0 \\ 0 & A_5^{[22]} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} = \begin{bmatrix} A_5^{[11]} & 0 \\ 0 & \Theta A_5^{[22]} \Theta \end{bmatrix},$$

$$\begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} B_5^{[11]} & B_5^{[12]} \\ B_5^{[12]} & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} = \begin{bmatrix} B_5^{[11]} & B_5^{[12]} \Theta \\ \Theta B_5^{[12]} & 0 \end{bmatrix}$$

and that

$$A_5^{[22]} = \text{diag}\{I_p - \Phi^2, I_{n-p} + \tilde{I}_{n-p}(\Psi^+)^2\}. \tag{42}$$

Assuming the generic condition that  $A_5^{[22]}$  is nonsingular in general, the choice of the diagonal matrix

$$\Theta := B_5^{[12]} A_5^{[22]-1} \tag{43}$$

is sufficient to restore the Lancaster structure. Through the sequence of pencils  $\lambda B_i - A_i$ ,  $i = 1, \dots, 5$ , all of which are isospectral to the original pencil  $\lambda B - A$ , the final step of scaling completes the construction of the equivalence transformation.

Our major result is stated below.

**Theorem 4.** *Given a quadratic pencil  $Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K$  with real-valued matrix coefficients, let columns of  $X$  and  $Y$  denote the right and the left eigenvectors of the linear pencil  $\lambda B - A$ , respectively, where  $A$  and  $B$  are defined in Eq. (12). Assume that*

- (1) *All eigenvalues are simple,*
- (2) *The matrix  $B_1$  defined in Eq. (19) is invertible,*
- (3) *The matrix  $A_5^{[22]}$  involved in Eq. (42) is invertible.*

Define

$$X^{[6]} := X^{[5]} \Gamma = X B_1^{-1/2} A^{-1/2} P J_r F \Gamma,$$

$$Y^{[6]} := Y^{[5]} \Gamma = Y B_1^{-1/2} A^{-1/2} P J_\ell F \Gamma, \tag{44}$$

where  $B_1$  is the diagonal matrix given by Eq. (19),  $P$  is the permutation matrix determined by Eq. (22),  $J_r$  and  $J_\ell$  are the realization matrices defined by Eq. (30),  $F$  is the elimination matrix defined by Eq. (33), and  $\Gamma$  is the scaling matrix given by Eq. (41). Then the original pencil  $\lambda B - A$  is isospectral to the pencil  $\lambda B_6 - A_6$ , where

$$A_6 := Y^{[6]\top} A X^{[6]} = \begin{bmatrix} -K_D & 0 \\ 0 & M_D \end{bmatrix},$$

$$B_6 := Y^{[6]\top} B X^{[6]} = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \tag{45}$$

and  $M_D$ ,  $C_D$  and  $K_D$  are real-valued diagonal matrices. In other words, the quadratic pencil  $\lambda^2 M + \lambda C + K$  is totally decoupled by isospectral transformations into  $\lambda^2 M_D + \lambda C_D + K_D$ .

The desirable structure preservation transformations  $\Pi_\ell$  and  $\Pi_r$  are given by

$$\Pi_\ell := Y^{[6]\top}, \quad \Pi_r := X^{[6]}, \tag{46}$$

which by our construction described above are real-valued.

### 3.4. Numerical example

We conclude this section with a numerical example. Up to this point, note that the theory is for general quadratic pencils where the matrix coefficients are general matrices in  $\mathbb{R}^{n \times n}$ . We emphasize that the construction of the equivalence transformation begins with the availability of the complete spectral information, the so called *Jordan triplet* in Ref. [9].

For the easy running of text, we report only four numeric digits even though all calculations involved are accurate to the machine accuracy. Consider the case

$$M = \begin{bmatrix} 0.7621 & 0.4447 & 0.7382 & 0.9169 \\ 0.4565 & 0.6154 & 0.1763 & 0.4103 \\ 0.0185 & 0.7919 & 0.4057 & 0.8936 \\ 0.8214 & 0.9218 & 0.9355 & 0.0579 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3710 & -1.0226 & 0.3155 & 0.5045 \\ 0.7283 & 1.0378 & 1.5532 & 1.8645 \\ 2.1122 & -0.3898 & 0.7079 & -0.3398 \\ -1.3573 & -1.3813 & 1.9574 & -1.1398 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.2111 & -0.6014 & -2.0046 & 1.2366 \\ 1.1902 & 0.5512 & -0.4931 & -0.6313 \\ -1.1162 & -1.0998 & 0.4620 & -2.3252 \\ 0.6353 & 0.0860 & -0.3210 & -1.2316 \end{bmatrix}.$$

Its spectrum in ascending order of moduli is given by

$$\{0.0348, -0.0877, 0.5453, -2.0162 \pm 1.4196i, 0.1815 \pm 2.5083i, 3.0425\},$$

among which the four real eigenvalues are classified into

$$C_d = \{3, 8\}, \quad C_f = \{2, 1\}.$$

Be aware of the order in  $C_f$  because the eigenvector corresponding to the eigenvalue 0.0348 is purely imaginary. The structure preserving transformation matrices  $\Pi_\ell$  and  $\Pi_r$  are given by

$$\Pi_\ell = \begin{bmatrix} 0.4189 & 0.9173 & 0.0945 & -0.4584 & -2.0559 & -1.9378 & -0.0730 & 1.2765 \\ -6.9347 & 12.4232 & -0.9207 & -2.4408 & 3.2216 & -7.7792 & -14.4484 & -8.8834 \\ -0.5439 & 0.0331 & 1.9456 & -2.3711 & -0.4656 & -0.4208 & 2.0679 & -1.4933 \\ -0.3465 & -10.1901 & -3.9018 & 12.2692 & -0.0115 & 0.0231 & -0.0563 & 0.0023 \\ 0.3381 & 0.3187 & 0.0120 & -0.2100 & -0.9446 & -0.3678 & 0.0461 & 0.3881 \\ -0.5094 & 1.2300 & 2.2846 & 1.4046 & -7.1196 & 12.8698 & -0.0912 & -1.9307 \\ 0.2807 & 0.2536 & -1.2465 & 0.9001 & 0.4631 & 0.9430 & -2.5265 & 0.8583 \\ -3.7728 & 7.5521 & -18.4169 & 0.7416 & -0.1468 & -10.5898 & -2.9272 & 12.2299 \end{bmatrix},$$

$$\Pi_r = \begin{bmatrix} 0.3340 & -3.7885 & 0.5893 & -8.7045 & 0.0300 & 0.9596 & -0.8007 & -8.7108 \\ -0.2226 & -14.6629 & 1.0132 & 12.2692 & -0.0866 & 3.9812 & -0.2769 & 0.7416 \\ -0.8990 & 10.1808 & -2.4537 & -4.3424 & -0.4661 & 2.5332 & 1.0515 & -13.8562 \\ 0.9328 & 6.0803 & -1.6896 & -2.5127 & 0.3928 & -3.4286 & 0.8571 & -9.4260 \\ -0.1822 & -6.0687 & 1.3284 & -0.0266 & 0.2132 & -3.4401 & -2.2834 & -8.2435 \\ 0.5264 & -25.1784 & 0.4594 & 0.0023 & 0.1265 & -13.2174 & 0.0197 & 12.2299 \\ 2.8341 & -16.0210 & -1.7445 & -0.0423 & 0.9806 & 11.1005 & 1.3190 & -3.6091 \\ -2.3881 & 21.6836 & -1.4220 & -0.0288 & -0.6509 & 4.8354 & 1.3855 & -2.0139 \end{bmatrix}.$$

The totally decoupled system has diagonal matrix coefficients given by

$$M_D = \text{diag}\{-0.4463, -120.8951, -2.7605, 457.1029\},$$

$$C_D = \text{diag}\{-1.7998, 43.8949, 9.9041, 24.1913\},$$

$$K_D = \text{diag}\{-2.7138, -764.5811, -4.5797, -1.3969\}.$$

If so desired, further scaling by  $\pm 1$  can change the diagonal entries of  $M_D$  into nonnegative number. Regardless, the five single-degree-of-freedom subsystems are easy to read by now.

#### 4. Self-adjoint pencil and congruence transformation

Thus far, we have shown that a general quadratic pencil  $Q(\lambda; M, C, K)$  can be totally decoupled by means of equivalence transformation. In this section, we argue further that a self-adjoint quadratic pencil, that is, when the matrix coefficients  $M$ ,  $C$  and  $K$  are all symmetric, can be totally decoupled by congruence transformations.

Observer first that we may choose both left and right eigenvectors of the symmetric pencil  $\lambda B - A$  to be the same. Thus we begin with  $X = Y$  in Eq. (18). Let  $(\lambda, \mathbf{x})$  denote a real-valued eigenpair for the pencil  $\lambda B - A$ . Note that

$$\lambda = \frac{\mathbf{x}^\top A \mathbf{x}}{\mathbf{x}^\top B \mathbf{x}}. \tag{47}$$

Our goal in this section is to show that the real eigenvalues can be regrouped in such a way that all pairs lie within categories  $C_a$ ,  $C_b$  and  $C_c$  only. This assertion, claimed by Garvey et al., has never been proved in Ref. [4].

The key ingredient in our construction of a congruence transformation is the notion of factorization for self-adjoint matrix polynomials [6, Chapter 11]. The machinery used to develop the general theory for matrix polynomials of arbitrary degree is quite involved. For simplicity and for completeness, we outline its idea for quadratic pencils via the following theorem. The main thrust is on the grouping of real eigenvalues into two disjoint subsets.

**Theorem 5.** *Assume that the leading coefficient  $M$  in the self-adjoint quadratic pencil  $Q(\lambda; M, C, K)$  is nonsingular and that the spectrum is simple. Let  $(\lambda, \mathbf{x}(\lambda))$  denote a real-valued eigenpair for the associated pencil  $\lambda B - A$  defined in Eq. (12). Then, for almost all self-adjoint pencils  $Q(\lambda; M, C, K)$ , the real eigenvalues can be divided evenly into two groups: half of the eigenvalues are such that the product  $\mathbf{x}(\lambda)^\top B \mathbf{x}(\lambda) > 0$  and the other half are such that  $\mathbf{x}(\lambda)^\top B \mathbf{x}(\lambda) < 0$ .*

**Proof.** Using Eq. (18), we already know  $B_1 = Y^\top B X$  is a diagonal matrix. Write

$$B_1 = \text{diag}\{b^{[1]}, \dots, b^{[2n]}\}.$$

Similar to what we have done in Section 3.1, let  $\mathcal{C}$ ,  $\mathcal{P}$  and  $\mathcal{N}$  denote, respectively, the index sets of complex-valued, positive, and negative diagonal entries in  $B_1$ . Write

$$\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-,$$

where  $\mathcal{C}^+$  and  $\mathcal{C}^-$  are the index subsets of diagonal entries of  $B_1$  with positive and negative imaginary parts, respectively. Again, let elements in  $\mathcal{C}$  be denoted by the lower case  $\mathbf{c}_j$  and so on. We want to show that  $|\mathcal{C}^+ \cup \mathcal{P}| = |\mathcal{C}^- \cup \mathcal{N}| = n$ .

One of the two sets  $\mathcal{C}^+ \cup \mathcal{P}$  and  $\mathcal{C}^- \cup \mathcal{N}$  will have cardinality at least  $n$ . We may assume, without loss of generality, that  $|\mathcal{C}^+ \cup \mathcal{P}| \geq n$ . The other case can be argued similarly. Denote  $\rho_1 := |\mathcal{C}^+|$  and  $\rho_2 := |\mathcal{P}|$ . Consider the matrix

$$Z := X_{\mathcal{C}^+ \cup \mathcal{P}} = [\mathbf{x}_{\mathbf{c}_1^+}, \dots, \mathbf{x}_{\mathbf{c}_{\rho_1}^+}, \mathbf{x}_{\mathbf{p}_1}, \dots, \mathbf{x}_{\mathbf{p}_{\rho_2}}] \in \mathbb{C}^{2n \times \rho_1} \oplus \mathbb{R}^{2n \times \rho_2},$$

whose columns are selected from those of  $X$  with column indices given by  $\mathcal{C}^+ \cup \mathcal{P}$ . By assumption, the matrix  $Z$  is of full column rank  $\rho_1 + \rho_2$ . Using the relationship Eq. (14), write

$$Z = \begin{bmatrix} U \\ UY \end{bmatrix},$$

where

$$U := U_{\mathcal{C}^+ \cup \mathcal{D}} = [\mathbf{u}_{\mathcal{C}_1^+}, \dots, \mathbf{u}_{\mathcal{C}_{\rho_1}^+} \mathbf{u}_{\rho_1}, \dots, \mathbf{u}_{\rho_2}] \in \mathbb{C}^{n \times \rho_1} \oplus \mathbb{R}^{n \times \rho_2} \tag{48}$$

and

$$Y := A_{\mathcal{C}^+ \cup \mathcal{D}} \tag{49}$$

denote the upper half of the matrix  $Z$  and the corresponding portion from the diagonal matrix  $A$  defined in Eq. (15), respectively. We want to show that  $U$  is of full column rank. If this is true, then because we have assumed  $\rho_1 + \rho_2 \geq n$ , it must be  $\rho_1 + \rho_2 = n$ .

To show the full rank of  $U$ , we know by construction that  $Z^\top BZ$  is a diagonal matrix with some complex-valued entries. By the facts that  $X^\top BZ$  is diagonal and that  $\bar{X}_{\mathcal{C}^+} = X_{\mathcal{C}^-}$ , it follows that the matrix

$$D := Z^* BZ = \text{diag}\{0, \dots, 0, b^{[\rho_1]}, \dots, b^{[\rho_2]}\},$$

where  $Z^*$  denotes the transpose of the complex conjugate of  $Z$ , has nonnegative entries since  $b^{[y]} > 0$  by definition. Suppose  $U\mathbf{z} = 0$  for some  $\mathbf{z} \in \mathbb{C}^{\rho_1 + \rho_2}$ . It is easy to see that

$$\mathbf{z}^* D \mathbf{z} = [0 \quad (UY\mathbf{z})^*] \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} 0 \\ UY\mathbf{z} \end{bmatrix} = 0.$$

It follows that  $D\mathbf{z} = 0$ . This is equivalent to

$$U^* M U Y \mathbf{z} = 0 \tag{50}$$

and, hence  $(UY\mathbf{z})^* M U Y \mathbf{z} = 0$ .

The set of symmetric matrices  $(M, C, K)$  such that

$$\{\mathbf{u} \in \mathbb{C}^{n \times n} \mid \mathbf{u}^* M \mathbf{u} = 0\} \cap \{UY\mathbf{z} \in \mathbb{C}^{n \times n} \mid \mathbf{z} \in \mathbb{C}^{\rho_1 + \rho_2}, MUY^2 + CUY + KU = 0\} \neq \emptyset$$

forms an algebraic variety in the space of  $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ . This variety has measure zero. That is, for almost all self-adjoint quadratic pencils  $Q(\lambda; M, C, K)$ , the condition Eq. (50) implies that  $UY\mathbf{z} = 0$ . We have thus proved that  $\mathbf{z}$  is in the null space of  $Z$ . Since  $Z$  is of full column rank, it must be  $\mathbf{z} = 0$  and, hence,  $U$  is of full column rank.

We remark that in Ref. [9] these two sets  $\mathcal{C}^+ \cup \mathcal{D}$  and  $\mathcal{C}^- \cup \mathcal{N}$  are said to be  $B$ -nonnegative and  $B$ -nonpositive, respectively. What we have proved about the factorization for almost all regular self-adjoint quadratic pencils is a special case of the more general decomposition theorem [6, Theorem 3.21].  $\square$

#### 4.1. With positive definite coefficients

One of the most important quadratic pencils in application is the class when  $M$  is symmetric and positive definite and both  $C$  and  $K$  are symmetric and positive semidefinite. It is known in this case that all eigenvalues of  $Q(\lambda)$  lie in the left half-plane. In particular, all real eigenvalues are nonpositive. The following result therefore is of great consequence in practice.

**Theorem 6.** *Under the same assumptions as in Theorem 5, almost all self-adjoint pencils  $Q(\lambda; M, C, K)$  with positive definite matrix coefficients can be totally decoupled by congruence transformations.*

**Proof.** It suffices to show that  $|\text{Re}| = |\text{Im}|$ . If this is true, then all real-valued eigenvalues are categorized into the single group  $C_a$ . It follows that  $L^+ = R^+$ ,  $L^- = R^-$  and, hence,  $J_\ell = J_r$ . In other words, the equivalence transformation involved in Eq. (45) becomes congruence transformation because  $Y^{[6]} = X^{[6]}$ .

The classification of  $\text{Re}$  and  $\text{Im}$  is based on the signs of the products  $\mathbf{x}^\top A \mathbf{x}$  where  $\mathbf{x}$  is a real-valued eigenvector associated with a real-valued eigenvalue. By Theorem 5, we have already known that precisely half of these eigenvectors are such that  $\mathbf{x}^\top B \mathbf{x}$  is positive and the other half eigenvectors give rise to negative products. Since all real-valued eigenvalues are of one sign, by Eq. (47) we know that precisely half of these eigenvectors are such that  $\mathbf{x}^\top A \mathbf{x}$  is positive and the other half leads to negative  $\mathbf{x}^\top A \mathbf{x}$ . This shows  $|\text{Re}| = |\text{Im}|$ .  $\square$

#### 4.2. Without positive definite coefficients

There is no specific pattern for real-valued eigenvalues of a general self-adjoint quadratic pencil. However, for any given real-valued eigenpair  $(\lambda, \mathbf{x})$ , the signs of  $\lambda$ ,  $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ , and  $\mathbf{x}^\top \mathbf{B} \mathbf{x}$  are related through the relationship

$$\text{sgn}(\mathbf{x}^\top \mathbf{A} \mathbf{x}) = \text{sgn}(\lambda) \text{sgn}(\mathbf{x}^\top \mathbf{B} \mathbf{x}). \quad (51)$$

This relationship together with Theorem 5 enables us to refine the classification described in Section 3.1 as follows:

*Case 1:* Suppose that  $\rho = \min\{|\text{Re}|, |\text{Im}|\} \neq 0$ . Assume  $\lambda_{r_s} > 0$  for a certain  $s$ . Then necessarily  $\mathbf{x}_{r_s}^\top \mathbf{A} \mathbf{x}_{r_s} > 0$ . By Eq. (51), we must have the corresponding  $\mathbf{x}_{r_s}^\top \mathbf{B} \mathbf{x}_{r_s} > 0$ . By definition, we always have  $\mathbf{x}_{i_j}^\top \mathbf{A} \mathbf{x}_{i_j} < 0$  among all possible eigenvalues  $\lambda_{i_j}$ . By Theorem 5, it cannot be so that  $\mathbf{x}_{i_j}^\top \mathbf{B} \mathbf{x}_{i_j} > 0$  for all  $j$ . There must be a certain  $t$  such that  $\mathbf{x}_{i_t}^\top \mathbf{B} \mathbf{x}_{i_t} < 0$ . By Eq. (51) again, we have the corresponding  $\lambda_{i_t} > 0$ . We therefore can pair  $\{r_s, r_t\} \in C_a$ . Likewise, a given  $\lambda_{r_s} < 0$  can be paired with a certain  $\lambda_{i_t} < 0$ . Repeating this argument, we see that  $C_f = \emptyset$  since no pairs of eigenvalues should be placed in  $C_f$ . Only  $C_a$  is needed and contains  $\rho$  pairs.

*Case 2:* After we have exhausted as many pairs in  $C_a$  as possible, suppose there are some leftovers in  $\text{Re}$  but none in  $\text{Im}$ . Denote this remaining subset of  $\text{Re}$  by  $\widehat{\text{Re}}$ . Note that  $\mathbf{x}_{r_j}^\top \mathbf{A} \mathbf{x}_{r_j} > 0$  for all  $r_j \in \widehat{\text{Re}}$ . Assume that one such eigenvalue  $\lambda_{r_s} > 0$ . Then  $\mathbf{x}_{r_s}^\top \mathbf{B} \mathbf{x}_{r_s} > 0$ . However, by Theorem 5, it cannot be so that  $\mathbf{x}_{r_j}^\top \mathbf{B} \mathbf{x}_{r_j} > 0$  for all the remaining  $r_j \in \widehat{\text{Re}}$ . There must be a certain  $r_t \in \widehat{\text{Re}}$  such that  $\mathbf{x}_{r_t}^\top \mathbf{B} \mathbf{x}_{r_t} < 0$ . By Eq. (51), we must have the corresponding  $\lambda_{r_t} < 0$ . Therefore, the pair  $(r_s, r_t)$  is categorized into  $C_b$ . Likewise, a given  $\lambda_{r_s} < 0$  from the remaining subset  $\widehat{\text{Re}}$  can be paired with a certain  $\lambda_{r_t} > 0$  with  $r_t \in \widehat{\text{Re}}$ . Repeating this argument, we see that no pairs of eigenvalues from  $\widehat{\text{Re}}$  should be placed in  $C_d$ . Only  $C_b$  is needed and  $C_d = \emptyset$ .

*Case 3:* Suppose that Case 2 does not happen. Rather, suppose that after we have exhausted as many pairs in  $C_a$  as possible, there are some leftovers in  $\text{Im}$  but none in  $\text{Re}$ . Denote this remaining subset of  $\text{Im}$  by  $\widehat{\text{Im}}$ . Note that  $\mathbf{x}_{i_j}^\top \mathbf{A} \mathbf{x}_{i_j} < 0$  for all  $i_j \in \widehat{\text{Im}}$ . Assume that one such eigenvalue  $\lambda_{i_s} > 0$ . Then  $\mathbf{x}_{i_s}^\top \mathbf{B} \mathbf{x}_{i_s} < 0$ . However, by Theorem 5, it cannot be so that  $\mathbf{x}_{i_j}^\top \mathbf{B} \mathbf{x}_{i_j} < 0$  for all the remaining  $i_j \in \widehat{\text{Im}}$ . There must be a certain  $i_t \in \widehat{\text{Im}}$  such that  $\mathbf{x}_{i_t}^\top \mathbf{B} \mathbf{x}_{i_t} > 0$ . By Eq. (51), we must have the corresponding  $\lambda_{i_t} < 0$ . Therefore, the pair  $(i_s, i_t)$  is categorized into  $C_c$ . Likewise, a given  $\lambda_{i_s} < 0$  from the remaining subset  $\widehat{\text{Im}}$  can be paired with a certain  $\lambda_{i_t} > 0$  with  $i_t \in \widehat{\text{Im}}$ . Repeating this argument, we see that no pairs of eigenvalues from  $\widehat{\text{Im}}$  should be placed in  $C_e$ . Only  $C_c$  is needed and  $C_e = \emptyset$ .

By now, we have generalized Theorem 6 to the general self-adjoint quadratic pencil.

**Theorem 7.** *Almost all self-adjoint pencil  $Q(\lambda; M, C, K)$  with simple spectrum can be totally decoupled by congruence transformations.*

**Proof.** The arguments in the above three cases show that the real-valued eigenvalues can be regrouped in such a way that all pairs lie with the categories  $C_a$ ,  $C_b$  and  $C_c$  only. An examination of the structure defined in Eq. (25) shows that  $L^+ = R^+$ ,  $L^- = R^-$  and, hence,  $J_\ell = J_r$ .  $\square$

## 5. Conclusion

It is commonly accepted that there is no equivalence transformation that can simultaneously diagonalize the three coefficient matrices in a general quadratic pencil. Following the ideas by Garvey et al., however, we have shown that real-valued transformations that decouple the original  $n$ -degree-of-freedom system as the direct sum of  $n$ -single-degree-of-freedom subsystems do exist for almost all quadratic pencils. The turning point that makes this advancement is that the notion of diagonalization is replaced by the Lancaster structure.

Our contributions in this paper are threefold: we offer a complete mathematical procedure showing how a given spectral decomposition which usually is complex-valued can be converted into real-valued equivalence transformations. Our approach, particularly in the steps of elimination and scaling, is easier and more

straightforward than that suggested in Ref. [4]. In case of self-adjoint quadratic pencils, we employ the notion of matrix polynomial factorization to prove a conjecture in Ref. [4] that the equivalence transformation is indeed a congruence transformation. This fact has never been justified before.

A prerequisite of the maneuver outlined in this paper is the availability of a complete spectral decomposition of the original quadratic pencil, which makes the process somewhat less feasible in practice. To develop a more effective numerical method to realize such a decoupling process without knowing a priori the complete spectral decomposition is an interesting open problem.

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